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Matrix model approach to the $\mathcal{N} = 2$ U(N) gauge theory with matter in the fundamental representation

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Abstract: We use matrix model technology to study the $\mathcal{N} = 2$ U(N) gauge theory with N_f massive hypermultiplets in the fundamental representation. We perform a completely perturbative calculation of the periods a_i and the prepotential $\mathcal{F}(a)$ up to the first instanton level, finding agreement with previous results in the literature. We also derive the Seiberg-Witten curve and differential from the large-M solution of the matrix model. We show that the two cases $N_f < N$ and $N \leq N_f < 2N$ can be treated simultaneously.

Keywords: Nonperturbative Effects, 1/N Expansion, Extended Supersymmetry, Matrix Models.

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Contents

1. Introduction

Dijkgraaf, Vafa, and collaborators have discovered remarkable relations between perturbative matrix models and instanton effects in supersymmetric gauge theories [1]–[4]. Recently we used the new matrix model technology to study the $\mathcal{N} = 2$ U(N) gauge theory [5] (ref. [5] also contains a more extensive list of references). We calculated the prepotential $\mathcal{F}(a)$ and the periods a_i perturbatively up to the first instanton level. A new ingredient in our calculation was a completely perturbative definition of the periods a_i as functions of the classical moduli e_i. Our results combined with those of Dijkgraaf and Vafa show that, even when the matrix model cannot be completely solved, a perturbative diagrammatic expansion of the matrix model can still be used to obtain all the low-energy non-perturbative information of $\mathcal{N} = 2$ gauge theories order-by-order in the instanton expansion.

In this paper we study the $\mathcal{N} = 2$ U(N) gauge theory with N_f hypermultiplets transforming in the fundamental representation of the gauge group using matrix model techniques. Several new features present themselves in this case, making the model well worth studying.

In the first part of the paper, we extend the perturbative results obtained in [5] for the $\mathcal{N} = 2$ U(N) theory to the case with N_f fundamental matter hypermultiplets. A new feature of the calculation, compared to the one in [5], is the appearance of planar diagrams with boundaries [6]. These contribute, in the diagrammatic expansion of the matrix model, to the free energy and superpotential. They also affect the relation between the periods a_i and the classical moduli e_i. We compute the periods a_i and prepotential $\mathcal{F}(a)$ perturbatively to first order in the instanton expansion, finding agreement with earlier results in the literature. This agreement is a test of our proposed relation [5] between a_i and e_i.

In the case of $\mathrm{U}(N)$ with fundamental matter, there is an ambiguity in the form of the Seiberg-Witten curve [7] for $N \leq N_f < 2N$ [8]–[10], with different forms of the curve corresponding to different definitions of the classical moduli e_i. These different curves yield slightly different relations between a_i and e_i. Our perturbative calculation, which does not start from a curve, yields an unambiguous relation between a_i and e_i and therefore implies a particular form the of the Seiberg-Witten curve, which we show to be $y^2 = \prod_{i=1}^{N}(x-e_i)^2 - f_{N-1}(x)$ where $f_{N-1}(x)$ is an $(N-1)$th order polynomial specified in eq. (6.3).

In the second part of the paper we derive the form of the Seiberg-Witten curve and differential for the $\mathcal{N}=2$ $\mathrm{U}(N)$ gauge theory with N_f fundamental hypermultiplets, from the large-M saddle-point solution to the matrix model, without any additional input. The result is consistent with known results [7]–[10] and also agrees with the form of the curve implied by the perturbative calculation. Our results give further support to the idea that all the low-energy information about the $\mathcal{N}=2$ theory is contained in the matrix model.[1] (Very recently some aspects of the relation between matrix models and Seiberg-Witten theory have been discussed in ref. [12].)

An important question is why the matrix model approach to supersymmetric gauge theories works and what the scope and limitations of the method are. Recently, these questions have been explored and purely field-theoretic proofs for the correctness of the matrix model approach have been presented for the pure $\mathcal{N}=1$ $\mathrm{U}(N)$ gauge theory with an arbitrary polynomial superpotential [13, 14]. It would be interesting to extend these results to cover the model studied in this paper. Also, ref. [15] discusses some aspects of the correspondence between matrix-model and gauge-theory quantities.

In section 2 we set up the perturbative calculation. In section 3 we calculate τ_{ij} as a function of the classical moduli to first order in the instanton expansion. In section 4 we extend our proposed perturbative definition of the periods a_i to the case when fundamentals are present, and use this result to determine the one-instanton corrections to a_i. In section 5 we compute the one-instanton correction to the prepotential $\mathcal{F}(a)$. When $N_f \geq N$ a certain polynomial appears in the relation between a_i and the classical moduli; the role of this polynomial is clarified in section 6. In section 7 we derive the Seiberg-Witten curve from the large-M saddle point solution to the matrix model. In section 8 we derive the Seiberg-Witten differential from within the matrix model framework. We conclude the paper with a summary of our findings.

[1] The matrix model also knows about string theory corrections in the form of curvature couplings [3]; some such couplings were recently computed [11] using matrix model techniques.

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2. Perturbative matrix model approach

In this section, we describe the perturbative matrix model approach to the $\mathcal{N}=2$ U(N) gauge theory with matter in the fundamental representation, extending our earlier work [5]. Previous work discussing matter in the fundamental representation (focusing mainly on $\mathcal{N}=1$ theories) in the matrix model context can be found in [6] and [16]–[20].

In the presence of (massless or massive) $\mathcal{N}=2$ hypermultiplets transforming in the fundamental representation, the $\mathcal{N}=2$ U(N) gauge theory develops a superpotential

$$W_{\rm mat}(\phi,q,\tilde{q}) = \sum_{I=1}^{N_f}\left[\tilde{q}_I\phi\, q^I + m_I\tilde{q}_I q^I\right], \tag{2.1}$$

written in terms of the $\mathcal{N}=1$ fields ϕ (the adjoint scalar in the $\mathcal{N}=2$ vector multiplet), q^I $(I=1,\dots N_f)$ transforming in the fundamental representation and $\tilde{q}_I$, transforming in the conjugate fundamental representation. We have suppressed the gauge group indices, and m_I are the masses of the fundamentals.

The first step of the matrix model program is to break $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$ by adding a tree-level superpotential $W_0(\phi)$ to the gauge theory. The particular choice of $W_0(\phi)$ relevant to us is the one that freezes the moduli to a generic point on the Coulomb branch of the $\mathcal{N}=2$ theory:

$$W_0(\phi) = \alpha\sum_{\ell=0}^{N}\frac{s_{N-\ell}(e)}{\ell+1}\,{\rm tr}(\phi^{\ell+1}) \quad\Rightarrow\quad W_0'(x) = \alpha\prod_{i=1}^{N}(x-e_i)\,, \tag{2.2}$$

where e_i are the classical moduli, $s_m(e)$ is the elementary symmetric polynomial

$$s_m(e) = (-1)^m\sum_{i_1<i_2<\cdots<i_m} e_{i_1}e_{i_2}\cdots e_{i_m}\,, \qquad s_0=1\,, \tag{2.3}$$

and α is a parameter that will be taken to zero at the end of the calculation, restoring $\mathcal{N}=2$ supersymmetry [21].

The next step is to reinterpret the superpotential $W(\phi,q,\tilde{q}) = W_0(\phi)+W_{\rm mat}(\phi,q,\tilde{q})$ as the potential of a chiral matrix model [1]–[4], which has the partition function (denoting the matrix model analogs of ϕ, q, and $\tilde{q}$ with capital letters)

$$Z = \frac{1}{{\rm vol}(G)}\int {\rm d}\Phi\,{\rm d}Q^I{\rm d}\tilde{Q}_I\exp\left(-\frac{W(\Phi,Q,\tilde{Q})}{g_s}\right), \tag{2.4}$$

where the integral is over $M\times M$ matrices Φ (which can be taken to be hermitean) and M-dimensional vectors Q^I and $\tilde{Q}_I$. In eq. (2.4), G is the unbroken matrix model gauge group, and g_s is a parameter that later will be taken to zero as $M\to\infty$. In taking the $M\to\infty$ limit, we keep N_f finite (as in ref. [18]); our approach thus differs from the one in [16]. The matrix integral (2.4) is evaluated perturbatively about an extremal point $\Phi=\Phi_0$, $Q_0=0$, $\tilde{Q}_0=0$ of $W(\Phi,Q,\tilde{Q})$. We write

$$\Phi=\Phi_0+\Psi=\begin{pmatrix} e_1\mathbb{1}_{M_1} & 0 & \cdots & 0\\ 0 & e_2\mathbb{1}_{M_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e_N\mathbb{1}_{M_N}\end{pmatrix}+\begin{pmatrix}\Psi_{11} & \Psi_{12} & \cdots & \Psi_{1N}\\ \Psi_{21} & \Psi_{22} & \cdots & \Psi_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ \Psi_{N1} & \Psi_{N2} & \cdots & \Psi_{NN}\end{pmatrix}, \tag{2.5}$$

where $\sum_i M_i = M$, and Ψ_{ij} is an $M_i \times M_j$ matrix. This choice breaks the U(M) symmetry to $G = \prod_{i=1}^{N} U(M_i)$.

The connected diagrams of the perturbative expansion of Z may be organized, using the standard double-line notation, in a topological expansion characterized by the Euler characteristic χ of the surface in which the diagram is embedded [22]

$$Z = \exp\left(\sum_{\chi\leq 2} g_s^{-\chi} F_\chi(e,S)\right) \quad \text{where} \quad S_i \equiv g_s M_i\,, \tag{2.6}$$

where $\chi = 2-2g-h$ with g the genus (number of handles) and h the number of holes. When evaluating the matrix integral in the $M_i \to \infty$, $g_s \to 0$ limit, with S_i held fixed, the leading contribution comes from the planar diagrams that can be drawn on the sphere $(\chi = 2)$,

$$F_{\rm s}(e,S) \equiv F_{\chi=2}(e,S) = g_s^2 \log Z\Big|_{\rm sphere}\,. \tag{2.7}$$

As discussed in [6], the presence of the Q^I, $\tilde{Q}_I$'s leads to the introduction of surfaces with boundaries in the topological expansion. The leading boundary contribution comes from surfaces with one boundary (disks), obtained from the sphere by cutting out one hole, and having $\chi = 1$,

$$F_{\rm d}(e,S) \equiv F_{\chi=1}(e,S) = g_s \log Z\Big|_{\rm disk}\,. \tag{2.8}$$

It was shown in [5] (generalizing the result in [4] for U(2)) that when one expands $W_0(\Phi)$ (2.2) to quadratic order in Ψ, the coefficients of $\mathrm{tr}(\Psi_{ij}\Psi_{ji})$ vanish when $i{\neq}j$. Hence the off-diagonal matrices Ψ_{ij} are zero modes, and correspond to pure gauge degrees of freedom. As in ref. [4, 5], we fix the gauge $\Psi_{ij} = 0$ $(i{\neq}j)$ and introduce Grassmann-odd ghost matrices B and C with action

$$\mathrm{tr}\,(B[\Phi,C]) = \sum_{i=1}^{N}\sum_{j\neq i}(e_i - e_j)\,\mathrm{tr}(B_{ji}C_{ij}) + \sum_{i=1}^{N}\sum_{j\neq i}\mathrm{tr}(B_{ji}\Psi_{ii}C_{ij} - B_{ji}C_{ij}\Psi_{jj})\,. \tag{2.9}$$

In the $\Psi_{ij} = 0$ $(i{\neq}j)$ gauge $W_0(\Phi)$ becomes [5]

$$W_0(\Phi) = \sum_{i=1}^{N} M_i W_0(e_i) + \alpha\sum_{i=1}^{N}\frac{R_i}{2}\,\mathrm{tr}(\Psi_{ii}^2) + \alpha\sum_{i=1}^{N}\sum_{p=3}^{N}\frac{\gamma_{p,i}}{p}\,\mathrm{tr}(\Psi_{ii}^p)\,, \tag{2.10}$$

where $R_i = \prod_{j\neq i} e_{ij}$ with $e_{ij} = e_i - e_j$, and

$$\gamma_{p,i} = \frac{1}{(p-1)!}\left[\left(\frac{\partial}{\partial x}\right)^{p-1}\prod_{k=1}^{N}(x-e_k)\right]\Bigg|_{x=e_i}\,. \tag{2.11}$$

Writing $Q^I = (Q_1^I, Q_2^I, \ldots, Q_N^I)^T$, where Q_i^I is an M_i-dimensional vector and similarly for $\tilde{Q}_I$, and expanding $W_{\rm mat}(\Phi, Q^I, \tilde{Q}_I)$ around the vacuum (2.5) one finds (using the $\Psi_{ij} = 0$ $(i{\neq}j)$ gauge)

$$W_{\rm mat}(\Phi, Q, \tilde{Q}) = \sum_{i=1}^{N}\sum_{I=1}^{N_f}\left[(e_i + m_I)\tilde{Q}_{iI}Q_i^I + \tilde{Q}_{iI}\Psi_{ii}Q_i^I\right]. \tag{2.12}$$

Collecting the above results, the partition function is given by the gauge-fixed integral

$$Z_{\text{g.f.}} = \frac{1}{\text{vol}(G)} \exp\left(-\frac{1}{g_s}\sum_{i=1}^{N} M_i W_0(e_i)\right) \int \mathrm{d}\Psi_{ii}\, \mathrm{d}B_{ij}\, \mathrm{d}C_{ij}\, \mathrm{d}Q^I \mathrm{d}\tilde{Q}_I \mathrm{e}^{I_{\text{quad}}+I_{\text{int}}}\,, \tag{2.13}$$

where the quadratic part of the action is

$$I_{\text{quad}} = -\frac{\alpha}{g_s}\sum_{i=1}^{N}\frac{R_i}{2}\,\text{tr}(\Psi_{ii}^2) - \sum_{i=1}^{N}\sum_{j\neq i} e_{ij}\,\text{tr}(B_{ji}C_{ij}) - \frac{1}{g_s}\sum_{i=1}^{N}\sum_{I=1}^{N_f}(e_i+m_I)\tilde{Q}_{iI}Q_i^I\,, \tag{2.14}$$

and the interaction terms are

$$I_{\text{int}} = -\frac{\alpha}{g_s}\sum_{i=1}^{N}\sum_{p=3}^{N}\frac{\gamma_{p,i}}{p}\,\text{tr}(\Psi_{ii}^p) - \sum_{i=1}^{N}\sum_{j\neq i}\text{tr}(B_{ji}\Psi_{ii}C_{ij} - B_{ji}C_{ij}\Psi_{jj}) - \frac{1}{g_s}\sum_{i=1}^{N}\sum_{I=1}^{N_f}\tilde{Q}_{iI}\Psi_{ii}Q_i^I\,. \tag{2.15}$$

The propagators for the various fields can be read off from eq. (2.14) and the vertices from eq. (2.15). Each ghost loop will acquire an additional factor of (-2) [4].

3. Perturbative calculation of $\tau_{ij}(e)$

The integral over the part of the quadratic action (2.14) involving Ψ_{ii}, B_{ij}, and C_{ij} can be explicitly performed [5]; including also the classical piece one finds (up to an e_i-independent quadratic monomial in the S_i's)

$$F_{\text{s}}(e,S) = -\sum_{i=1}^{N} S_i W_0(e_i) + \frac{1}{2}\sum_{i=1}^{N} S_i^2 \log\left(\frac{S_i}{\alpha R_i \hat{\Lambda}^2}\right) + \sum_{i=1}^{N}\sum_{j\neq i} S_i S_j \log\left(\frac{e_{ij}}{\hat{\Lambda}}\right) + \sum_{n\geq 3} F_{\text{s}}^{(n)}(e,S)\,. \tag{3.1}$$

As in [4], we have included in eq. (3.1) a contribution $-\left(\sum_{i=1}^{N} S_i\right)^2 \log\hat{\Lambda}$ that reflects the ambiguity in the cut-off of the full $\mathrm{U}(M)$ gauge group. (A similar contribution is included in (3.4) below.) The term $F_{\text{s}}^{(n)}(e,S)$ is an nth order polynomial in S_i arising from planar loop diagrams built from the interaction vertices [3]. The contribution to $F_{\text{s}}(e,S)$ cubic in S_i was computed in [5] with the result:

$$\begin{aligned}\alpha F_{\text{s}}^{(3)}(e,S) = {}& (\frac{1}{2}+\frac{1}{6})\sum_i \frac{S_i^3}{R_i}\left(\sum_{k\neq i}\frac{1}{e_{ik}}\right)^2 - \frac{1}{4}\sum_i\frac{S_i^3}{R_i}\sum_{k\neq i}\sum_{\ell\neq i,k}\frac{1}{e_{ik}e_{i\ell}} - \\ & -2\sum_i\sum_{k\neq i}\frac{S_i^2 S_k}{R_i e_{ik}}\sum_{\ell\neq i}\frac{1}{e_{i\ell}} + 2\sum_i\sum_{k\neq i}\sum_{\ell\neq i}\frac{S_i S_k S_\ell}{R_i e_{ik} e_{i\ell}} - \sum_i\sum_{k\neq i}\frac{S_i^2 S_k}{R_i e_{ik}^2}\,.\end{aligned} \tag{3.2}$$

Next we turn to the contribution of the fundamentals Q, $\tilde{Q}$ to the matrix model free energy. Since these involve quark loops, they only contribute to the disk-level part of the free energy. The integral over the quadratic part of W_{mat} gives[2]

$$\int \prod_{i=1}^{N}\prod_{I=1}^{N_f} \mathrm{d}Q_i^I \mathrm{d}\tilde{Q}_{iI} \exp\left(-\frac{1}{g_s}(e_i+m_I)\tilde{Q}_{iI}Q_i^I\right) = \exp\left(-\sum_{i=1}^{N}\sum_{I=1}^{N_f} M_i \log\frac{(e_i+m_I)}{g_s}\right) \tag{3.3}$$

[2]Note that there are no $M_i \log M_i$ terms in the expansion of $1/\text{vol}(G)$ [23].

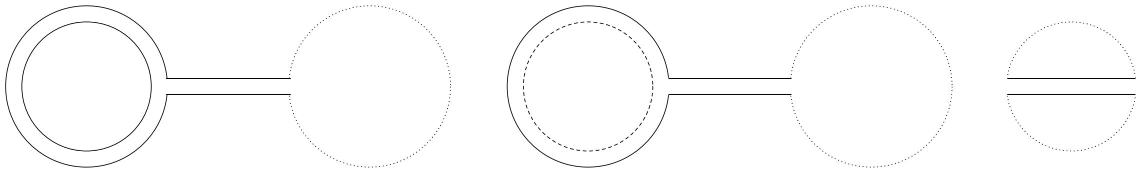

Figure 1: Disk diagrams contributing to $F_{\rm d}(e,S)$ at order $\mathcal{O}(S^2)$. Solid double lines refer to Ψ_{ii} propagators, solid-plus-dashed double lines refer to ghost propagators, and single dotted lines correspond to the propagator for the Q's.

which yields (up to an e_i-independent part linear in S_i) the first term of

$$F_{\rm d}(e,S) = -\sum_{i=1}^{N}\sum_{I=1}^{N_f} S_i \log\frac{(e_i+m_I)}{\hat{\Lambda}} + \sum_{n\geq 2} F_{\rm d}^{(n)}(e,S) \tag{3.4}$$

Here $F_{\rm d}^{(n)}(e,S)$ is an nth order polynomial in S_i arising from planar disk diagrams built from the interaction vertices. To obtain the $\mathcal{O}(S^2)$ contribution to $F_{\rm d}(e,S)$, we need to evaluate the diagrams displayed in figure 1.

One might also consider diagrams drawn on surfaces with additional holes. One example is a "dumb-bell" diagram as in figure 1, but with quark propagators at both ends. Such a diagram corresponds to a sphere with two holes, the dotted lines encircling each of the two holes. However, such a surface has $\chi = 0$ and the diagram is therefore suppressed by a factor of g_s relative to the $\chi = 1$ disk contribution in the $g_s \to 0$, $M_i \to \infty$ limit.

The above diagrams lead to:

$$\alpha F_{\rm d}^{(2)}(e,S) = \sum_{I=1}^{N_f}\left[\sum_i \frac{S_i^2}{R_i f_{iI}}\sum_{j\neq i}\frac{1}{e_{ij}} - 2\sum_i\sum_{j\neq i}\frac{S_iS_j}{R_i e_{ij} f_{iI}} + \frac{1}{2}\sum_i \frac{S_i^2}{R_i f_{iI}^2}\right], \tag{3.5}$$

where $f_{iI} = e_i + m_I$.

To relate the matrix model and its free energy to the $\mathcal{N} = 2$ U(N) gauge theory (with N_f hypermultiplets in the fundamental representation of the gauge group) broken to $\prod_i \mathrm{U}(N_i)$, one introduces [1]–[3], [24] and [6]

$$W_{\rm eff}(e,S) = -\sum_{i=1}^{N} N_i \frac{\partial F_{\rm s}(e,S)}{\partial S_i} - F_{\rm d}(e,S) + 2\pi i \tau_0 \sum_{i=1}^{N} S_i\,, \tag{3.6}$$

where $\tau_0 = \tau(\Lambda_0)$ is the gauge coupling of the U(N) theory at some scale Λ_0. Since we are breaking U(N) to U$(1)^N$, we set $N_i = 1$ for $i = 1,\ldots,N$. It was conjectured in ref. [6] that the disk-level part of the free energy $F_{\rm d}(e,S)$ contributes to $W_{\rm eff}$ without any derivatives acting on it. We will find further support for this claim. Next, one extremizes the effective superpotential with respect to S_i to obtain $\langle S_i\rangle$:

$$\left.\frac{\partial W_{\rm eff}(e,S)}{\partial S_i}\right|_{S_j=\langle S_j\rangle} = 0\,. \tag{3.7}$$

Finally,

$$\tau_{ij}(e) = \frac{1}{2\pi i}\frac{\partial^2 F_{\rm s}(e,S)}{\partial S_i \partial S_j}\bigg|_{S_i=\langle S_i\rangle} \tag{3.8}$$

yields the couplings of the unbroken $\mathrm{U}(1)^N$ factors of the gauge theory, as a function of e_i. Note that although both the Seiberg-Witten formula $\tau_{ij}(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j}$ and (3.8) refer to the same quantity (the period matrix of the Seiberg-Witten curve Σ), they are expressed in terms of different parameters on the moduli space (a_i vs. e_i).

Above, we have evaluated $F_{\rm s}(e,S)$ to cubic order in S_i and $F_{\rm d}(e,S)$ to quadratic order in S_i, which will be sufficient to obtain $\tau_{ij}(e)$ to one-instanton accuracy. Inserting the results eq. (3.1), (3.2), (3.4), and (3.5) in eq. (3.6), we obtain

$$\begin{aligned} W_{\rm eff}(e,S) =& \sum_i W_0(e_i) - \sum_i S_i \log\left(\frac{S_i}{\alpha R_i \hat{\Lambda}^2}\right) - 2\sum_i\sum_{k\neq i} S_k \log\left(\frac{e_{ik}}{\hat{\Lambda}}\right) + \\ &+ \sum_i \sum_{I=1}^{N_f} S_i \log\left(\frac{f_{iI}}{\hat{\Lambda}}\right) - \frac{1}{\alpha}\left[\sum_i\sum_{k\neq i}\sum_{\ell\neq i,k}\left(-\frac{3S_i^2}{4R_i e_{ik}e_{i\ell}} + \frac{2S_k S_\ell}{R_i e_{ik} e_{i\ell}}\right) + \right. \\ &\qquad + \sum_i\sum_{k\neq i}\left(-\frac{S_i^2}{R_i e_{ik}^2} - \frac{2S_iS_k}{R_i e_{ik}^2} + \frac{2S_i^2}{R_k e_{ik}^2}\right) + \\ &\qquad + \sum_{I=1}^{N_f}\sum_i\left(-\frac{S_i^2}{R_i f_{iI}}\sum_{k\neq i}\frac{1}{e_{ik}} + \sum_{k\neq i}\frac{2S_iS_k}{R_i e_{ik} f_{iI}} - \right. \\ &\qquad \left.\left. - \frac{S_i^2}{2R_i f_{iI}^2}\right)\right] + (2\pi i \tau_0 + \mathrm{const})\sum_i S_i \,. \end{aligned} \tag{3.9}$$

The extrema $\langle S_i\rangle$ are obtained from (3.7), and can be evaluated in an expansion in Λ

$$\begin{aligned} \langle S_i\rangle =& \frac{\alpha L_i}{R_i}\Lambda^{2N-N_f} + \frac{\alpha L_i}{R_i}\Lambda^{4N-2N_f}\times \\ &\times\left[\sum_{k\neq i}\sum_{\ell\neq i,k}\left(\frac{3L_i}{2R_i^2 e_{ik}e_{i\ell}} + \frac{4L_\ell}{R_k R_\ell e_{ik} e_{k\ell}}\right) + \right. \\ &+ \sum_{k\neq i}\left(\frac{2L_i}{R_i^2 e_{ik}^2} - \frac{4L_i}{R_i R_k e_{ik}^2} + \frac{2L_k}{R_i R_k e_{ik}^2} + \frac{2L_k}{R_k^2 e_{ik}^2}\right) - \sum_{I=1}^{N_f}\frac{L_i}{R_i^2 f_{iI}^2} + \\ &\left. + \sum_{I=1}^{N_f}\sum_{k\neq i}\left(-\frac{2L_i}{R_i^2 e_{ik} f_{iI}} + \frac{2L_k}{R_i R_k e_{ik} f_{iI}} - \frac{2L_k}{R_k^2 e_{ik} f_{kI}}\right)\right] + \mathcal{O}(\Lambda^{6N-3N_f})\,. \end{aligned} \tag{3.10}$$

where $L_i = \prod_{I=1}^{N_f}(e_i + m_I)$, and various constants as well as τ_0 have been absorbed into a redefinition of the cut-off, $\Lambda = \mathrm{const}\times\hat{\Lambda}\,\mathrm{e}^{\pi i \tau_0/N}$.

Although we are primarily interested in the $\mathcal{N}=2$ limit in this paper, the $\mathcal{N}=1$ effective superpotential may be easily computed by substituting eq. (3.10) into eq. (3.9). In the case $N_f \geq N$ one has to proceed with care, see [16, 18, 19] for further details.

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Below we will make repeated use of the identity

$$\sum_{k\neq i}\frac{L_k}{R_k e_{ik}}=-\frac{L_i}{R_i}\sum_{k\neq i}\frac{1}{e_{ik}}+\frac{L_i}{R_i}\sum_I\frac{1}{f_{iI}}-\tilde{T}(e_i) \tag{3.11}$$

which can be derived by taking the $z\to e_i$ limit of both sides of

$$\frac{\prod_{I=1}^{N_f}(z+m_I)}{\prod_{k=1}^{N}(z-e_k)}-\frac{L_i}{R_i(z-e_i)}=\sum_{k\neq i}\frac{L_k}{R_k(z-e_k)}+\tilde{T}(z)\,. \tag{3.12}$$

where the polynomial $\tilde{T}(z)=\sum_{k=0}^{N_f-N}\tilde{t}_k z^{N_f-N-k}$ is the positive part of the Laurent expansion of $\prod_{I=1}^{N_f}(z+m_I)/\prod_{k=1}^{N}(z-e_k)$ and is only non-zero when $N_f\geq N$. More explicitly, the coefficients $\tilde{t}_k$ are exactly as in [25, eqs. (2.4) and (2.5)]; note that our e_i are the same as their $\bar{a}_i$.

We can now evaluate

$$\tau_{ij}(e)=\frac{1}{2\pi i}\frac{\partial^2 F_{\mathrm{s}}(e,S)}{\partial S_i\partial S_j}\bigg|_{S_i=\langle S_i\rangle}=\tau_{ij}^{\mathrm{pert}}(e)+\sum_{d=1}^{\infty}\Lambda^{(2N-N_f)d}\tau_{ij}^{(d)}(e)\,. \tag{3.13}$$

The perturbative contribution (up to an additive constant) is

$$2\pi i\tau_{ij}^{\mathrm{pert}}(e)=\delta_{ij}\left[-\sum_{k\neq i}\log\left(\frac{e_{ik}}{\Lambda}\right)^2+\sum_{I=1}^{N_f}\log\left(\frac{f_{iI}}{\Lambda}\right)\right]+(1-\delta_{ij})\left[\log\left(\frac{e_{ij}}{\Lambda}\right)^2\right]. \tag{3.14}$$

Using the identity (3.11) one obtains the one-instanton contribution

$$\begin{aligned}2\pi i\tau_{ij}^{(1)}(e)=\delta_{ij}\Bigg[&\sum_{k\neq i}\sum_{\ell\neq i,k}\left(\frac{8L_i}{R_i^2e_{ik}e_{i\ell}}-\frac{4L_k}{R_k^2e_{ik}e_{k\ell}}\right)+\sum_{k\neq i}\left(\frac{10L_i}{R_i^2e_{ik}^2}+\frac{10L_k}{R_k^2e_{ik}^2}+\frac{4\tilde{T}(e_i)}{R_ie_{ik}}-\frac{4\tilde{T}(e_k)}{R_ke_{ik}}\right)+\\&+\sum_{I=1}^{N_f}\left(\frac{L_i}{R_i^2f_{iI}^2}+\sum_{J\neq I}\frac{2L_i}{R_i^2f_{iI}f_{iJ}}-\sum_{k\neq i}\frac{8L_i}{R_i^2e_{ik}f_{iI}}+\sum_{k\neq i}\frac{2L_k}{R_k^2e_{ik}f_{kI}}-\frac{2\tilde{T}(e_i)}{R_if_{iI}}\right)\Bigg]+\\&+(1-\delta_{ij})\Bigg[\sum_{k\neq i,j}\left(-\frac{8L_i}{R_i^2e_{ij}e_{ik}}-\frac{8L_j}{R_j^2e_{ji}e_{jk}}+\frac{4L_k}{R_k^2e_{ik}e_{jk}}\right)-\frac{10L_i}{R_i^2e_{ij}^2}-\frac{10L_j}{R_j^2e_{ij}^2}+\\&+\sum_{I=1}^{N_f}\left(\frac{4L_i}{R_i^2e_{ij}f_{iI}}+\frac{4L_j}{R_j^2e_{ji}f_{jI}}\right)-\frac{4\tilde{T}(e_i)}{R_ie_{ij}}-\frac{4\tilde{T}(e_j)}{R_je_{ji}}\Bigg]\end{aligned} \tag{3.15}$$

to the gauge coupling matrix. Finally, we take the limit $\alpha\to 0$ to restore $\mathcal{N}=2$ supersymmetry, but this has no effect on τ_{ij}, which is independent of α.

4. Perturbative determination of a_i

If we are to use the matrix model results (3.14) and (3.15) to determine the $\mathcal{N}=2$ prepotential $\mathcal{F}(a)$, we must first express τ_{ij} in terms of the periods a_i. In [5] we proposed a definition of a_i within the context of the perturbation expansion of the matrix model, without referring to the Seiberg-Witten curve or differential.[3] We argued in [5] that a_i can be determined perturbatively via

$$a_i = \left.\frac{\partial \tilde{W}^i_{\rm eff}(e,\langle\tilde{S}\rangle,\epsilon)}{\partial\epsilon}\right|_{\epsilon\to 0}, \tag{4.1}$$

where $\tilde{W}^i_{\rm eff}(e,S,\epsilon)$ is the effective superpotential that one obtains by considering the matrix model with action $\tilde{W}^i(\Phi,Q,\tilde{Q}) = W(\Phi,Q,\tilde{Q}) + \epsilon\ {\rm tr}_i\ \Phi$. Here the trace is only over the ith block. For motivations for this proposal we refer the reader to [5]. In the present case, it is sufficient to consider

$$\begin{aligned}\tilde{Z}^i &= \frac{1}{{\rm vol}(G)}\int {\rm d}\Phi \exp\left(-\frac{1}{g_s}\left[W(\Phi,Q,\tilde{Q}) + \epsilon\ {\rm tr}_i\ \Phi\right]\right)\\ &= \exp\left(\frac{1}{g_s^2}\tilde{F}^i_{\rm s}(e,S,\epsilon) + \frac{1}{g_s}\tilde{F}^i_{\rm d}(e,S,\epsilon) + \cdots\right).\end{aligned} \tag{4.2}$$

Writing $\tilde{F}^i_{\rm s}(e,S,\epsilon) = F_{\rm s}(e,S) + \epsilon\delta F^i_{\rm s}$ and similarly for $\tilde{F}^i_{\rm d}(e,S,\epsilon)$, and observing that to first order in ϵ

$$\tilde{Z}^i = Z + \frac{1}{{\rm vol}(G)}\int {\rm d}\Phi\left[-\frac{\epsilon}{g_s}\right]{\rm tr}_i\ \Phi \exp\left(-\frac{W(\Phi,Q,\tilde{Q})}{g_s}\right), \tag{4.3}$$

one finds $\delta F^i_{\rm s} = -g_s\langle{\rm tr}_i\ \Phi\rangle|_{\rm sphere}$, where $\langle{\rm tr}_i\ \Phi\rangle|_{\rm sphere}$ is obtained by calculating all connected one-point functions at sphere-level in the matrix model with action $W(\Phi,Q,\tilde{Q})$. Similarly, $\delta F^i_{\rm d} = -\langle{\rm tr}_i\ \Phi\rangle|_{\rm disk}$ where $\langle{\rm tr}_i\ \Phi\rangle|_{\rm disk}$ is obtained by computing all connected one-point functions at disk-level.

Now the effective potential for the matrix integral (4.2) is

$$\begin{aligned}\tilde{W}^i_{\rm eff}(e,S,\epsilon) &= -\sum_{j=1}^N N_j\frac{\partial\tilde{F}^i_{\rm s}(e,S,\epsilon)}{\partial S_j} - \tilde{F}^i_{\rm d}(e,S,\epsilon) + 2\pi i\tau_0\sum_{j=1}^N S_j\\ &= W_{\rm eff}(e,S) - \epsilon\left[\sum_{j=1}^N N_j\frac{\partial}{\partial S_j}\delta F^i_{\rm s} + \delta F^i_{\rm d}\right].\end{aligned} \tag{4.4}$$

Extremizing $\tilde{W}^i_{\rm eff}(e,S,\epsilon)$ with respect to S gives $\langle\tilde{S}_i\rangle = \langle S_i\rangle + \epsilon\delta S_i + \mathcal{O}(\epsilon^2)$. Substituting $\langle\tilde{S}\rangle$ into eq. (4.4), one obtains

$$\tilde{W}^i_{\rm eff}(e,\langle\tilde{S}\rangle,\epsilon) = W_{\rm eff}(e,\langle S\rangle) + \epsilon\sum_{j=1}^N\delta S_j\left.\frac{\partial W_{\rm eff}}{\partial S_i}\right|_{\langle S\rangle} - \epsilon\left.\left[\sum_{j=1}^N N_j\frac{\partial}{\partial S_j}\delta F^i_{\rm s} + \delta F^i_{\rm d}\right]\right|_{\langle S\rangle} + \mathcal{O}(\epsilon^2). \tag{4.5}$$

The second term vanishes by the definition of $\langle S\rangle$. Finally, using eq. (4.1), one obtains

$$a_i = -\left.\left[\sum_{j=1}^N N_j\frac{\partial}{\partial S_j}\delta F^i_{\rm s} + \delta F^i_{\rm d}\right]\right|_{\langle S\rangle}. \tag{4.6}$$

[3] For the model studied in this paper the Seiberg-Witten curve is known [7]–[10] and the relationship between a_i and e_i is straightforwardly obtained [25] from the A_i-period integral. However, our goal in this section is to determine a_i using only the matrix model perturbation expansion.

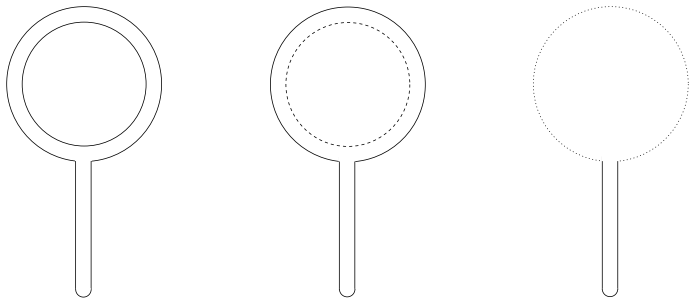

Figure 2: Tadpole diagrams contributing to the one-instanton contribution to a_i.

Considering a generic point in moduli space, where $\mathrm{U}(N) \to \mathrm{U}(1)^N$ (so that $N_i = 1$) and expanding Φ around the vacuum (2.5), $\mathrm{tr}_i\ \Phi = M_i e_i + \mathrm{tr}(\Psi_{ii})$, we find

$$a_i = e_i + \left[\sum_{j=1}^{N} \frac{\partial}{\partial S_j} g_s \langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{sphere}} + \langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{disk}} \right] \Bigg|_{\langle S \rangle}, \tag{4.7}$$

where $\langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{sphere}}$ is obtained by calculating, using the matrix model (2.13), all connected planar tadpole diagrams with an external Ψ_{ii} leg that can be drawn on a sphere, and $\langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{disk}}$ is obtained by computing all connected planar tadpole diagrams with an external Ψ_{ii} leg at disk-level in the topological expansion.

It should be emphasized that (4.7) offers a completely perturbative means of obtaining the relation between a_i and e_i, which does not require knowledge of the Seiberg-Witten curve or differential.

We will now evaluate eq. (4.7) for the case of the $\mathcal{N} = 2$ $\mathrm{U}(N)$ gauge theory with N_f fundamental hypermultiplets. The relevant tadpole diagrams contributing to first order in the instanton expansion are displayed in figure 2.

The first two diagrams contribute to $\langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{sphere}}$. These were evaluated in [5] with the result

$$\langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{sphere}} = \frac{1}{\alpha g_s} \sum_{j \neq i} \left[-\frac{S_i^2}{R_i e_{ij}} + 2 \frac{S_i S_j}{R_i e_{ij}} \right]. \tag{4.8}$$

The third diagram in figure 2 contributes to $\langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{disk}}$. By using the Feynman rules derived from the action (2.13) one finds

$$\langle \mathrm{tr}\ \Psi_{ii} \rangle|_{\mathrm{disk}} = -\frac{S_i}{\alpha R_i} \sum_{I=1}^{N_f} \frac{1}{f_{iI}}. \tag{4.9}$$

Inserting the above results into eq. (4.7), evaluating the resulting expression using eq. (3.10), and using the identity (3.11), one finds

$$a_i = e_i + \Lambda^{2N-N_f} \left(-\frac{2L_i}{R_i^2} \sum_{j \neq i} \frac{1}{e_{ij}} + \frac{L_i}{R_i^2} \sum_{I=1}^{N_f} \frac{1}{f_{iI}} - \frac{2\tilde{T}(e_i)}{R_i} \right) + \mathcal{O}(\Lambda^{4N-2N_f}). \tag{4.10}$$

The relation between a_i and e_i that we have just derived agrees precisely, at the one-instanton level, with [25, eq. (3.10)], provided that the polynomial $T(x)$ in their expression is set equal to $\frac{1}{2}\tilde{T}(x)$. We will discuss the implications of this result in section 6.

5. Perturbative calculation of $\tau_{ij}(a)$ and $\mathcal{F}(a)$

Now that we have determined the relation between a_i and e_i, we can rewrite $\tau_{ij}(e)$ in terms of a_i, and from that determine the form of the prepotential $\mathcal{F}(a)$ to one-instanton accuracy. Equation (4.10) implies that

$$\begin{aligned}\log e_{ij} \;=\; \log a_{ij} + \Lambda^{2N-N_f}\Bigg[\sum_{k\neq i,j}\left(\frac{2L_i}{R_i^2 e_{ij}e_{ik}}+\frac{2L_j}{R_j^2 e_{ji}e_{jk}}\right)+\frac{2L_i}{R_i^2 e_{ij}^2}+\frac{2L_j}{R_j^2 e_{ij}^2}-\\ -\sum_{I=1}^{N_f}\left(\frac{L_i}{R_i^2 e_{ij}f_{iI}}+\frac{L_j}{R_j^2 e_{ji}f_{jI}}\right)+\frac{2\tilde{T}(e_i)}{R_i e_{ij}}+\frac{2\tilde{T}(e_j)}{R_j e_{ji}}\Bigg], \qquad (5.1)\end{aligned}$$

where $a_{ij} = a_i - a_j$, and

$$\log f_{iI} = \log(a_i + m_I) + \Lambda^{2N-N_f}\left[\sum_{k\neq i}\frac{2L_i}{R_i^2 e_{ik}f_{iI}} - \frac{L_i}{R_i^2 f_{iI}}\sum_{J=1}^{N_f}\frac{1}{f_{iJ}} + \frac{2\tilde{T}(e_i)}{R_i f_{iI}}\right]. \qquad (5.2)$$

We can now re-express τ_{ij} (3.15) in terms of a_i

$$\tau_{ij}(a) = \tau_{ij}^{\mathrm{pert}}(a) + \sum_{d=1}^{\infty}\Lambda^{(2N-N_f)d}\tau_{ij}^{(d)}(a)\,, \qquad (5.3)$$

where the perturbative contribution is (up to additive constants)

$$2\pi i\tau_{ij}^{\mathrm{pert}}(a) = \delta_{ij}\left[-\sum_{k\neq i}\log\left(\frac{a_i-a_k}{\Lambda}\right)^2+\sum_{I=1}^{N_f}\log\left(\frac{a_i+m_I}{\Lambda}\right)\right]+(1-\delta_{ij})\left[\log\left(\frac{a_i-a_j}{\Lambda}\right)^2\right] \qquad (5.4)$$

and the one-instanton contribution is

$$\begin{aligned}2\pi i\tau_{ij}^{(1)}(a) \;=\; \delta_{ij}\Bigg[\sum_{k\neq i}\left(\frac{4L_i}{R_i^2}\sum_{\ell\neq i,k}\frac{1}{a_{ik}a_{i\ell}}+\frac{6L_i}{R_i^2 a_{ik}^2}+\frac{6L_k}{R_k^2 a_{ik}^2}\right)+\\ +\sum_{I=1}^{N_f}\left(-\sum_{k\neq i}\frac{4L_i}{R_i^2 a_{ik}f_{iI}}+\sum_{J\neq I}\frac{L_i}{R_i^2 f_{iI}f_{iJ}}\right)\Bigg]+\\ +(1-\delta_{ij})\Bigg[\sum_{k\neq i,j}\left(-\frac{4L_i}{R_i^2 a_{ij}a_{ik}}-\frac{4L_j}{R_j^2 a_{ji}a_{jk}}+\frac{4L_k}{R_k^2 a_{ik}a_{jk}}\right)-\frac{6L_i}{R_i^2 a_{ij}^2}-\frac{6L_j}{R_j^2 a_{ij}^2}+\\ +\sum_{I=1}^{N_f}\left(\frac{2L_i}{R_i^2 a_{ij}f_{iI}}+\frac{2L_j}{R_j^2 a_{ji}f_{jI}}\right)\Bigg], \qquad (5.5)\end{aligned}$$

where now $R_i = \prod_{j\neq i}(a_i - a_j)$ and $f_{iI} = a_i + m_I$. Observe that all the $\tilde{T}(x)$ terms cancel out in the final expression for $\tau_{ij}(a)$. As will be discussed in more detail in the next section, $\tilde{T}(x)$ can be absorbed into a redefinition of the e_i [25]. Since $\tau_{ij}(a)$ is independent of e_i it should be insensitive to this redefinition, and therefore to the form of $\tilde{T}(x)$.

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Finally, it is readily verified that (5.4), (5.5) can be written as $\tau_{ij} = \partial^2 \mathcal{F}(a)/\partial a_i \partial a_j$ with (up to a quadratic polynomial)

$$2\pi i \mathcal{F}(a) = -\frac{1}{4}\sum_i \sum_{j\neq i}(a_i - a_j)^2 \log\left(\frac{a_i - a_j}{\Lambda}\right)^2 + \frac{1}{4}\sum_i \sum_{I=1}^{N_f}(a_i + m_I)^2 \log\left(\frac{a_i + m_I}{\Lambda}\right)^2 + \\ +\Lambda^{2N-N_f}\sum_i \prod_{j\neq i}\prod_{I=1}^{N_f}\frac{(a_i+m_I)}{(a_i-a_j)^2} + \mathcal{O}(\Lambda^{4N-2N_f})\,. \tag{5.6}$$

This precisely agrees with the result obtained in [25, eq. (4.34)].

To conclude, we have shown that a completely perturbative matrix model calculation, which does not use the Seiberg-Witten curve or differential, gives the correct result for the prepotential to first order in the instanton expansion for the U(N) gauge theory with N_f fundamentals. Higher-instanton corrections to the prepotential may be obtained by higher-loop contributions to the matrix model free energy and tadpole diagrams.

6. The meaning of $\tilde{T}(x)$

In ref. [25] D'Hoker, Krichever, and Phong derived the prepotential for the $\mathcal{N} = 2$ U(N) theory with N_f flavors from a Seiberg-Witten curve of the form[4]

$$y^2 = \left[\prod_{i=1}^{N}(x-e_i) + 4\Lambda^{2N-N_f}T(x)\right]^2 - 4\Lambda^{2N-N_f}\prod_{I=1}^{N_f}(x+m_I)\,. \tag{6.1}$$

In their work the $(N_f - N)$th order polynomial $T(x)$ was left unspecified (although two different candidates [8, 10] were presented) since, as shown in section 2.c of that paper, the prepotential $\mathcal{F}(a)$ is independent of $T(x)$. This is because $T(x)$ can always be absorbed into a redefinition of the e_i, and $\mathcal{F}(a)$ is insensitive to a redefinition of e_i. However, since $T(x)$ is tied to the definition of e_i, its form will affect the relation between a_i and e_i.

Our matrix model calculation of the relation between a_i and e_i (4.10) implies (via [25, eq. (3.10)]) a specific form for $T(x)$, namely

$$T(x) = \frac{1}{2}\tilde{T}(x) + \mathcal{O}(\Lambda^{2N-N_f}) = \frac{1}{2}\sum_{k=0}^{N_f-N}\tilde{t}_k x^{N_f-N-k} + \mathcal{O}(\Lambda^{2N-N_f})\,, \tag{6.2}$$

and thus corresponds to a specific choice of the e_i. (Our perturbative matrix model calculation only yields a result valid to one-instanton accuracy.) The Seiberg-Witten curve (6.1) corresponding to eq. (6.2) has the form

$$\begin{aligned} y^2 &= \prod_{i=1}^{N}(x-e_i)^2 - f(x)\,, \\ f(x) &= 4\Lambda^{2N-N_f}\left(\prod_{I=1}^{N_f}(x+m_I) - \tilde{T}(x)\prod_{i=1}^{N}(x-e_i)\right) + \mathcal{O}(\Lambda^{4N-2N_f})\,. \end{aligned} \tag{6.3}$$

[4]Note: Λ^{2N-N_f} in ref. [25] differs from ours by a factor of 4, except in eq. (4.34). In the e-print version of ref. [25] the factor of 4 in eq. (2.6) should be omitted, and the right hand sides in eq. (2.8) should be multiplied by 1/4. These typos are corrected in the published version.

The definition of $\tilde{T}(x)$, given below eq. (3.12), ensures that $f(x)$ is at most an $(N-1)$th order polynomial. Thus, the choice of e_i in the matrix model is such that none of the coefficients of x^N or higher powers in y^2 receive $O(\Lambda^{2N-N_f})$ corrections. (However, as we discuss below, the gauge-invariants $\langle u_n \rangle$ do receive corrections.) As we will see in the next section, this is exactly what the saddle-point solution of the matrix model requires.

It is curious to note that the form of $T(x)$ proposed in ref. [10] and on the right hand side of eq. (2.8) in ref. [25] is[5] $T(x) = (1/4)\sum_{k=0}^{N_f-N} \tilde{t}_k x^{N_f-N-k}$, precisely one-half of that in eq. (6.2). Why the difference?

Consider the gauge-invariant variables $\langle u_n \rangle = (1/n)\langle \mathrm{tr}(\phi^n)\rangle$, which classically have the values $(u_n)_{\mathrm{cl}} = (1/n)\sum_{i=1}^N e_i^n$. Quantum mechanically, these may be computed via [4, 5] $\langle u_n \rangle = (1/2\pi i n)\sum_{i=1}^N \oint_{A_i} x^{n-1}\lambda_{SW}$, where λ_{SW} is the Seiberg-Witten differential. They may also be computed in the matrix model [5], starting from the correlators $\langle \mathrm{tr}(\Phi^n)\rangle$ (and modifying the expressions of ref. [5] to include the $\langle \mathrm{tr}(\Phi^n)\rangle|_{\mathrm{disk}}$ contribution, as in eq. (4.7) of this paper; see section 8). It is easily shown that for $N_f < N$ (in which case $T(x)$ vanishes) $\langle u_n \rangle = (u_n)_{\mathrm{cl}}$ for $n = 1, \ldots, N$ [21, 5]. When $N_f \geq N$, however, $\langle u_n \rangle$ with $2N - N_f \leq n \leq N$ can get $\mathcal{O}(\Lambda^{2N-N_f})$ corrections.

As stated above, choosing a particular $T(x)$ corresponds to a particular choice of parameters e_i used to parametrize the moduli space. It is possible to define the N parameters e_i so that the relation $\langle u_n \rangle = (1/n)\sum_{i=1}^N e_i^n$ continues to hold quantum mechanically for $n = 1, \ldots, N$. This requirement then leads to the form of $T(x)$ in ref. [10, 25] (see however ref. [26]). In contrast, for the choice of $T(x)$ in eq. (6.2), $\langle u_n \rangle = (u_n)_{\mathrm{cl}}$ no longer holds at the one-instanton level.

7. Matrix model derivation of the Seiberg-Witten curve

In this section, we will derive the form of the Seiberg-Witten curve for $\mathcal{N}=2$ U(N) gauge theory with $N_f < 2N$ fundamental hypermultiplets by solving the matrix model integral using saddle-point methods (for a review of this method, see, e.g., ref. [27]).

Our starting point is the matrix model partition function (2.4)

$$Z = \frac{1}{\mathrm{vol}(G)}\int \mathrm{d}\Phi\, \mathrm{d}Q^I \mathrm{d}\tilde{Q}_I \exp\left(-\frac{1}{g_s}W_0(\Phi) - \frac{1}{g_s}\sum_{I=1}^{N_f}\left[\tilde{Q}_I \phi Q^I + m_I \tilde{Q}_I Q^I\right]\right). \tag{7.1}$$

Diagonalizing Φ and integrating over $Q, \tilde{Q}$, one obtains (λ_i are the eigenvalues of Φ) [1, 16]

$$Z \propto \int \prod_{i=1}^{M} \mathrm{d}\lambda_i \exp\left(-\frac{1}{g_s}\sum_i W_0(\lambda_i) + 2\sum_{i<j}\log(\lambda_i - \lambda_j) - \sum_{I=1}^{N_f}\sum_i \log(\lambda_i + m_I)\right). \tag{7.2}$$

The saddle-point equation is obtained by varying the action with respect to λ_i:

$$-\frac{1}{g_s}W_0'(\lambda_i) + 2\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}\sum_{I=1}^{N_f}\frac{1}{\lambda_i + m_I} = 0\,. \tag{7.3}$$

[5]Taking into account the correction in the previous footnote 4.

To solve (7.3), it is standard procedure [27] to introduce the trace of the resolvent

$$\omega(x) = \frac{1}{M}\,\mathrm{tr}\left(\frac{1}{\Phi - x}\right) = \frac{1}{M}\sum_i \frac{1}{\lambda_i - x} \tag{7.4}$$

which can be shown to satisfy [27]

$$\begin{aligned} \omega^2(x) + \frac{W_0'(x)}{g_s M}\,\omega(x) + \frac{1}{g_s M^2}\sum_i \frac{W_0'(x) - W_0'(\lambda_i)}{x - \lambda_i} - \\ -\frac{1}{M}\omega'(x) + \frac{1}{M^2}\sum_i \sum_{I=1}^{N_f} \frac{1}{(\lambda_i - x)(\lambda_i + m_I)} = 0\,. \end{aligned} \tag{7.5}$$

Now we let $g_s \to 0$, $M \to \infty$, with $S = g_s M$ held fixed. We also hold N_f fixed; in this, our approach differs from ref. [16]. In this limit, the last two terms of eq. (7.5) vanish.

The large-M limit expressions are conveniently written in terms of the density of eigenvalues

$$\rho(\lambda) = \frac{1}{M}\sum_i \delta(\lambda - \lambda_i)\,, \qquad \int \rho(\lambda)\,\mathrm{d}\lambda = 1\,. \tag{7.6}$$

In this language the resolvent becomes

$$\omega(x) = \int \mathrm{d}\lambda \frac{\rho(\lambda)}{\lambda - x}\,, \qquad \rho(\lambda) = \frac{1}{2\pi i}\left[\omega(\lambda + i\epsilon) - \omega(\lambda - i\epsilon)\right] \tag{7.7}$$

and eq. (7.5) can be rewritten as

$$\omega^2(x) + \frac{W_0'(x)}{S}\omega(x) + \frac{1}{4S^2}f(x) = 0\,, \tag{7.8}$$

where

$$f(x) = 4S\int \mathrm{d}\lambda\,\rho(\lambda)\,\frac{W_0'(x) - W_0'(\lambda)}{x - \lambda} \tag{7.9}$$

is an (as yet) arbitrary $(N-1)$th order polynomial. Defining

$$y(x) = 2S\omega(x) + W_0'(x) \tag{7.10}$$

one may rewrite eq. (7.8) as

$$y^2 = W_0'(x)^2 - f(x)\,, \qquad f(x) = \sum_{n=0}^{N-1} b_n x^n\,. \tag{7.11}$$

This equation characterizes a hyperelliptic Riemann surface. When the roots of $W_0'(x)$ are well-separated and $f(x)$ is a small correction to $W_0'(x)$, the curve has N cuts in the x plane, centered approximately on the roots of $W_0'(x)$. The eigenvalues of Φ are clustered along these cuts. The function $f(x)$ determines the distribution of the eigenvalues of Φ among the N cuts, and the spreading of those eigenvalues due to eigenvalue repulsion. Let M_i denote the number of eigenvalues along the i^{th} cut:

$$M_i = M\int_i \mathrm{d}\lambda\,\rho(\lambda)\,. \tag{7.12}$$

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Define $S_i = g_s M_i$, which remains finite in the M, $M_i \to \infty$ limit. Then, using eqs. (7.7) and (7.10), we see that eq. (7.12) may be rewritten

$$S_i = -\frac{1}{4\pi i} \oint_{A_i} y \, \mathrm{d}x \,, \tag{7.13}$$

where A_i denotes the contour surrounding the ith cut. This is eq. (3.10) of [1] (up to a factor of 2; the sign depends on the direction of the contour integrals, which we take to be counterclockwise). Up to this point, we have just been following ref. [1].

As in ref. [21], we denote by P and Q the points $x = \infty$ on the two sheets of the curve (7.11). (If one needs a cutoff for an integral, one takes P and Q to be at $x = \Lambda_0$ with Λ_0 large.) To be specific, let P be on the sheet on which $W_0'(x) - y(x)$ goes to zero as $x \to \infty$. Also, denote by C_i a path from Q to P that passes through the i^{th} cut. The Riemann surface of genus $N-1$ described by the curve (7.11) can be given a canonical homology basis as follows: A_i $(i = 1, \ldots, N-1)$ and $B_i = C_i - C_N$ $(i = 1, \ldots, N-1)$.

Our goal in the remainder of this section is to use matrix-model methods to determine the explicit form of $f(x)$ in the spectral curve (7.11). This will in turn yield the (hyper-elliptic) Seiberg-Witten curve for the U(N) theory with N_f fundamental hypermultiplets. The saddle-point evaluation of the partition function (7.2) gives (here we need to keep the first subleading term since it contributes to F_{d})

$$\begin{aligned} Z = \exp\Bigg(&-\frac{S}{g_s^2} \int \mathrm{d}\lambda \, \rho(\lambda) \, W_0(\lambda) + \frac{S^2}{g_s^2} \int \mathrm{d}\lambda \, \mathrm{d}\lambda' \, \rho(\lambda) \, \rho(\lambda') \log(\lambda - \lambda') - \\ &- \frac{S}{g_s} \sum_{I=1}^{N_f} \int \mathrm{d}\lambda \, \rho(\lambda) \, \log(\lambda + m_I) \Bigg) \end{aligned} \tag{7.14}$$

from which we infer

$$F_{\mathrm{s}} = -S \int \mathrm{d}\lambda \, \rho(\lambda) \, W_0(\lambda) + S^2 \int \mathrm{d}\lambda \, \mathrm{d}\lambda' \, \rho(\lambda) \, \rho(\lambda') \, \log(\lambda - \lambda') \tag{7.15}$$

and

$$F_{\mathrm{d}} = -S \sum_{I=1}^{N_f} \int \mathrm{d}\lambda \, \rho(\lambda) \, \log(\lambda + m_I) \,. \tag{7.16}$$

In order to compute W_{eff}, we need the variation of F_{s} under a small change in S_i. From (7.12) we see that such a variation can be implemented by letting $\rho(\lambda) \to \rho(\lambda) + (\delta S_i / S) \delta \times (\lambda - e_i)$ where e_i refers to an arbitrary, but fixed, point along the i^{th} cut. Using this result in (7.15) gives[6]

$$\delta F_{\mathrm{s}} = \delta S_i \left[-W_0(e_i) + 2S \int \mathrm{d}\lambda \, \rho(\lambda) \, \log(\lambda - e_i) \right] . \tag{7.17}$$

[6]See [1] and appendix A of ref. [28] for related discussions.

This may be rewritten as (here const refers to a constant of integration)

$$\begin{aligned}\frac{\partial F_{\rm s}}{\partial S_i} &= \int_{e_i}^{P} \mathrm{d}x\; W_0'(x) - 2S\int \mathrm{d}\lambda\,\rho(\lambda)\int_{e_i}^{P}\frac{\mathrm{d}x}{x-\lambda} + \mathrm{const}\\ &= \int_{e_i}^{P}\mathrm{d}x\,\big(W_0'(x) + 2S\,\omega(x)\big) + \mathrm{const}\\ &= \int_{e_i}^{P} y\,\mathrm{d}x + \mathrm{const}\end{aligned} \tag{7.18}$$

which is just [1, eq. (3.11)]. Using the fact that y differs only by a sign on the two sheets, together with the definition $B_i = C_i - C_N$, we may rewrite this as

$$\begin{aligned}\frac{\partial F_{\rm s}}{\partial S_i} &= \frac{1}{2}\int_{Q}^{e_i} y\,\mathrm{d}x + \frac{1}{2}\int_{e_i}^{P} y\,\mathrm{d}x + \mathrm{const}\\ &= \frac{1}{2}\int_{C_i} y\,\mathrm{d}x + \mathrm{const}\\ &= \frac{1}{2}\int_{B_i} y\,\mathrm{d}x + \frac{1}{2}\int_{C_N} y\,\mathrm{d}x + \mathrm{const}\,.\end{aligned} \tag{7.19}$$

For $W_{\rm eff}$, we will also need

$$\begin{aligned}F_{\rm d}(e,S) &= -S\sum_{I=1}^{N_f}\int \mathrm{d}\lambda\,\rho(\lambda)\int_{-m_I}^{P}\frac{\mathrm{d}x}{\lambda - x} + \mathrm{const}\\ &= -S\sum_{I=1}^{N_f}\int_{-m_I}^{P}\omega(x)\,\mathrm{d}x + \mathrm{const}\\ &= -\frac{1}{2}\sum_{I=1}^{N_f}\int_{-m_I}^{P} y(x)\,\mathrm{d}x + \mathrm{const}\,,\end{aligned} \tag{7.20}$$

where we absorb the S_i-independent $W_0(P)-W_0(-m_I)$ terms into the integration constant.

We now use eqs. (7.19) and (7.20) in the effective superpotential (setting $N_i = 1$)

$$\begin{aligned}W_{\rm eff} =& -\sum_{i=1}^{N}\frac{\partial F_{\rm s}}{\partial S_i} - F_{\rm d} + 2\pi i\tau_0\sum_{i=1}^{N} S_i \qquad (7.21)\\ =& -\frac{1}{2}\sum_{i=1}^{N-1}\oint_{B_i} y\,\mathrm{d}x - \frac{1}{2}N\int_{C_N} y\,\mathrm{d}x + \frac{1}{2}\sum_{I=1}^{N_f}\int_{-m_I}^{P} y\,\mathrm{d}x - \frac{1}{2}\tau_0\sum_{i=1}^{N}\oint_{A_i} y\,\mathrm{d}x + \mathrm{const}\,.\end{aligned}$$

In the prescription relating the matrix model and the $\mathcal{N}=2$ gauge theory we are instructed to extremize $W_{\rm eff}$ with respect to S_i. Since the S_i's are determined by $f(x)$ and therefore by the b_n's through eqs. (7.11) and (7.13), we may equivalently vary (7.21) with respect to b_n [21]. From eq. (7.11), one sees that $(\partial y/\partial b_n)\mathrm{d}x = -(1/2)x^n\mathrm{d}x/y$. For $0 \le n \le N-2$, these form a complete basis of holomorphic differentials on the Riemann surface [29]. We may therefore change basis to the unique basis of holomorphic differentials ζ_k dual to the homology basis, i.e., $\oint_{A_i} \zeta_k = \delta_{ik}$. Consequently, the equations $\delta W_{\rm eff}/\delta b_n = 0$ for $0 \le n \le N-2$ may be rewritten

$$0 = -\sum_{i=1}^{N-1} \oint_{B_i} \zeta_k - N \int_Q^P \zeta_k + \sum_{I=1}^{N_f} \int_{-m_I}^P \zeta_k \,, \tag{7.22}$$

where $\sum_{i=1}^N \oint_{A_i} \zeta_k = 0$ because the sum of A_i cycles is a trivial cycle. The first term just yields $\sum_{i=1}^{N-1} \tau_{ik}$, which is an element of the period lattice. Hence[7]

$$\begin{aligned} & N \int_P^Q \zeta_k + \sum_{I=1}^{N_f} \int_{-m_I}^P \zeta_k = \\ & = N \int_{p_0}^Q \zeta_k - (N-N_f) \int_{p_0}^P \zeta_k - \sum_{I=1}^{N_f} \int_{p_0}^{-m_I} \zeta_k = 0 \quad \text{(modulo the period lattice)} \,, \end{aligned} \tag{7.23}$$

where p_0 is an arbitrary (generic) point on the Riemann surface. It now follows from Abel's theorem [29] that there exists a function $\psi(x)$ on the Riemann surface with an Nth order pole at Q, an $(N-N_f)$th order zero (or pole, if $N_f > N$) at P, and simple zeros at $-m_I$ for $I = 1, \ldots, N_f$. As we will now show, this requirement suffices to fix the form of $f(x)$, and therefore the Seiberg-Witten curve.

For $0 \le N_f < N$, the function $\psi(x)$ is simply (proportional to) the resolvent:

$$\psi(x) = y - W_0'(x) = \sqrt{W_0'(x)^2 - f(x)} - W_0'(x), \qquad 0 \le N_f < N \,. \tag{7.24}$$

This can be seen as follows: $\psi(x)$ has an Nth order pole at Q, and (at least) a simple zero at P (because $f(x)$ is a polynomial of at most $(N-1)$th order). Abel's theorem yields $N-1$ conditions and therefore completely constrains the remaining zeros. Thus $\psi(x)$ must have a simple zero at $x = -m_I$, so $f(x)$ must contain a factor $(x + m_I)$ for each I. For $\psi(x)$ to have an $(N-N_f)$th order zero at P, $f(x)$ can be of N_fth order at most. These two conditions require $f(x) \propto \prod_{I=1}^{N_f}(x+m_I)$. Naming the constant of proportionality $4\Lambda^{2N-N_f}$, and setting $\alpha = 1$ in eq. (2.2), we see that the spectral curve (7.11) is given by

$$y^2 = \prod_{i=1}^N (x-e_i)^2 - 4\Lambda^{2N-N_f} \prod_{I=1}^{N_f} (x+m_I) \tag{7.25}$$

precisely the Seiberg-Witten curve [7]–[10] for $N_f < N$. (It should also be possible to determine this constant of proportionality by setting $\delta W_{\rm eff}/\delta b_{N-1} = 0$, and using the gauge theory relation $2\pi i \tau(\Lambda_0) = (2N - N_f)\log(\Lambda/\Lambda_0)$ and the fact that [21] $\sum_{i=1}^N \oint_{A_i} y \, dx = -\pi i b_{N-1}$.)

For $N \le N_f < 2N$, the function $\psi(x)$ is not given by the resolvent but by a related function

$$\psi(x) = \sqrt{A(x)^2 - g(x)} - A(x) \,, \qquad N \le N_f < 2N \,, \tag{7.26}$$

where $A(x)$ is an Nth order polynomial and $g(x) \propto \prod_{I=1}^{N_f}(x+m_I)$. (As before, we name the proportionality constant $4\Lambda^{2N-N_f}$.) Under these conditions, $\psi(x)$ vanishes at $x = -m_I$,

[7] This equation was obtained in ref. [21] for the case $N_f = 0$ by a somewhat different approach. Here we have derived it using only matrix-model methods.

for $I = 1, \dots N_f$, has an Nth order pole at Q, and an $(N_f - N)$th order pole at P. For $\psi(x)$ to be a function on the Riemann surface (7.11), the square root in $\psi(x)$ must be proportional to $y(x)$, that is (normalizing appropriately)

$$A(x)^2 - 4\Lambda^{2N-N_f} \prod_{I=1}^{N_f} (x + m_I) = W_0'(x)^2 - f(x)\,, \tag{7.27}$$

where $f(x)$ is a polynomial of order at most $(N-1)$. The solution to this, to $\mathcal{O}(\Lambda^{2N-N_f})$, is

$$\begin{aligned} A(x) &= \prod_{i=1}^{N} (x - e_i) + 2\Lambda^{2N-N_f} \tilde{T}(x)\,, \\ f(x) &= 4\Lambda^{2N-N_f} \left(\prod_{I=1}^{N_f} (x + m_I) - \tilde{T}(x) \prod_{i=1}^{N} (x - e_i) \right), \end{aligned} \tag{7.28}$$

where $\tilde{T}(x)$ is defined below eq. (3.12), and again we have set $\alpha = 1$ in eq. (2.2). Thus the spectral curve (7.11) and function $\psi(x)$ are given by

$$\begin{aligned} y^2 &= \prod_{i=1}^{N} (x - e_i)^2 - f(x)\,, \qquad N \le N_f < 2N \\ \psi(x) &= y - A(x)\,, \end{aligned} \tag{7.29}$$

in agreement with the Seiberg-Witten curve for $N \le N_f < 2N$ [7]–[10] but with a particular choice of subleading term $\tilde{T}(x)$. (This form of the curve was already obtained (6.3) in the previous section by comparing our perturbative matrix model calculation with the curve in ref. [25]. The subleading term $\tilde{T}(x)$ simply corresponds to a particular choice of moduli parameters e_i picked out by the matrix model.)

Thus, for both $N_f < N$ and $N \le N_f < 2N$, the spectral curve obtained from the matrix-model saddle-point integral agrees precisely with the known Seiberg-Witten curve (6.1) for the $\mathcal{N} = 2$ U(N) gauge theory with N_f fundamental hypermultiplets.

Finally, from the properties of $\psi(x)$ (7.24) and (7.26), we see that

$$h(x)\mathrm{d}x = \frac{\mathrm{d}\psi}{\psi} \tag{7.30}$$

is a meromorphic differential with simple poles at P, Q, and $x = -m_I$ and residues $N - N_f$, $-N$, and 1 respectively. These conditions imply that the meromorphic differential given by $\lambda_{SW} = x\, h(x)\mathrm{d}x$ has all the correct properties to be the Seiberg-Witten differential [7, 10, 30]. Moreover, using the specific forms of $\psi(x)$ given in eqs. (7.25) and (7.29), we obtain exactly the form of the λ_{SW} given in ref. [25].

8. Derivation of the Seiberg-Witten differential

In the previous section we obtained an expression (7.30) related to the Seiberg-Witten differential λ_{SW}. Although this form can be motivated from the Calabi-Yau approach [21, 31] it does not constitute a genuine matrix-model derivation of λ_{SW}. In this section we present a derivation of λ_{SW} entirely within the framework of the matrix model.

In the Seiberg-Witten approach, the gauge-theory expectation value of tr ϕ^n is calculated via [4, 5]

$$\langle \mathrm{tr}\ \phi^n \rangle = \frac{1}{2\pi i} \sum_{i=1}^{N} \oint_{A_i} x^{n-1} \lambda_{SW} \,. \tag{8.1}$$

The relation between the gauge theory vev and matrix model quantities is

$$\langle \mathrm{tr}\ \phi^n \rangle = \left[\sum_{j=1}^{N} \frac{\partial}{\partial S_j} g_s \langle \mathrm{tr}\ \Phi^n \rangle_{\mathrm{sphere}} + \langle \mathrm{tr}\ \Phi^n \rangle_{\mathrm{disk}} \right] \Bigg|_{\langle S \rangle} \tag{8.2}$$

which generalizes [5, eq. (5.10)] to the case when boundaries are present (see also [15]). The derivation of eq. (8.2) is similar to that of eq. (4.7) of this paper but uses the deformation $\tilde{W}(\Phi, Q, \tilde{Q}) = W(\Phi, Q, \tilde{Q}) + \epsilon\,(1/n)\ \mathrm{tr}\,(\Phi^n)$.

The matrix-model expectation values $\langle \mathrm{tr}\ \Phi^n \rangle$ in eq. (8.2) may be expressed in terms of the resolvent (7.4)

$$\begin{aligned} \omega(x) &= -\frac{1}{M} \left\langle \mathrm{tr}\, \frac{1}{x - \Phi} \right\rangle = -\frac{1}{M} \sum_{n=0}^{\infty} x^{-n-1} \langle \mathrm{tr}\, \Phi^n \rangle \\ \langle \mathrm{tr}\, \Phi^n \rangle &= -\frac{M}{2\pi i} \sum_{i=1}^{N} \oint_{A_i} x^n \omega(x) \mathrm{d}x \end{aligned} \tag{8.3}$$

which acts as a generating function for the expectation values. To proceed, we rewrite the last term in (7.5) as

$$\begin{aligned} \frac{1}{M^2} \sum_i \sum_{I=1}^{N_f} \frac{1}{(\lambda_i - x)(\lambda_i + m_I)} &= \frac{1}{M^2} \sum_i \frac{1}{\lambda_i - x} \sum_{I=1}^{N_f} \frac{1}{x + m_I} - \frac{1}{M^2} \sum_{I=1}^{N_f} \frac{1}{x + m_I} \sum_i \frac{1}{\lambda_i + m_I} \\ &= \frac{1}{M} \sum_{I=1}^{N_f} \frac{\omega(x) - \omega(-m_I)}{x + m_I} \end{aligned} \tag{8.4}$$

so that eq. (7.5) becomes

$$\begin{aligned} \omega^2(x) + \frac{W_0'(x)}{g_s M}\, \omega(x) + \frac{1}{g_s M^2} \sum_i \frac{W_0'(x) - W_0'(\lambda_i)}{x - \lambda_i} - & \\ -\frac{1}{M} \omega'(x) + \frac{1}{M} \sum_{I=1}^{N_f} \frac{\omega(x) - \omega(-m_I)}{x + m_I} &= 0 \,. \end{aligned} \tag{8.5}$$

Next, we expand $\omega(x)$ as

$$\omega(x) = \sum_{\chi \le 2} \frac{1}{M^{2-\chi}} \omega_{1-\chi/2}(x) = \omega_0(x) + \frac{1}{M} \omega_{1/2}(x) + \mathcal{O}\left(\frac{1}{M^2} \right) . \tag{8.6}$$

Using the method developed in ref. [32],[8] we can solve the loop-equation (8.5) order-by-order in $1/M$, which in principle will give us $\langle \mathrm{tr}\ \Phi^n \rangle$ to arbitrary order in the topological

[8] See also the recent paper [33].

expansion. For eq. (8.2), however, we will only need $\omega_{\rm s}(x) \equiv \omega_0(x)$ and $\omega_{\rm d}(x) \equiv \omega_{1/2}(x)$. Inserting (8.6) into eq. (8.5), and using the fact [32][9] that the $(1/M)\omega'(x)$ term is $\mathcal{O}(1/M^2)$, we find

$$\begin{aligned} \omega_{\rm s}(x) &= \frac{1}{2S}\left[y - W_0'(x)\right] , \\ \omega_{\rm d}(x) &= -\frac{S}{y}\sum_{I=1}^{N_f}\frac{\omega_{\rm s}(x)-\omega_{\rm s}(-m_I)}{x+m_I} , \end{aligned} \tag{8.7}$$

where $y^2 = W_0'(x)^2 - f(x)$. This result, together with (8.3), allows us to write the contributions to $\langle {\rm tr}\ \Phi^n\rangle$ at the sphere ($\chi = 2$) and disk ($\chi = 1$) levels as

$$\begin{aligned} \langle {\rm tr}\ \Phi^n\rangle_{\rm sphere} &= -\frac{M}{2\pi i}\sum_{i=1}^{N}\oint_{A_i} x^n\, \omega_{\rm s}(x)\, {\rm d}x , \\ \langle {\rm tr}\ \Phi^n\rangle_{\rm disk} &= -\frac{1}{2\pi i}\sum_{i=1}^{N}\oint_{A_i} x^n\, \omega_{\rm d}(x)\, {\rm d}x . \end{aligned} \tag{8.8}$$

Inserting these expressions into eq. (8.2) and comparing with (8.1) one can read off

$$\lambda_{SW} = x\left[\sum_{i=1}^{N}\frac{\partial}{\partial S_i}(-S\omega_{\rm s}(x)) - \omega_{\rm d}(x)\right]\Bigg|_{\langle S\rangle} {\rm d}x . \tag{8.9}$$

This generalizes eq. (5.3) in v3 of ref. [15] to the case when boundaries are present.

Using eq. (8.7), we have

$$\sum_{i=1}^{N}\frac{\partial}{\partial S_i}(-S\omega_{\rm s}(x)) = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial y}{\partial S_i} . \tag{8.10}$$

This expression has unit A_i-periods,

$$\frac{1}{2\pi i}\oint_{A_i}\left[-\frac{1}{2}\sum_{j=1}^{N}\frac{\partial y}{\partial S_j}\right]{\rm d}x = \sum_{j=1}^{N}\frac{\partial}{\partial S_j}\left[-\frac{1}{4\pi i}\oint_{A_i} y\,{\rm d}x\right] = \sum_{j=1}^{N}\frac{\partial}{\partial S_j}S_i = 1 \tag{8.11}$$

using the definition of S_i (7.13). Moreover, by writing (b_n was defined in eq. (7.11))

$$\begin{aligned} -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial y}{\partial S_i} = -\frac{1}{2}\sum_{i=1}^{N}\sum_{n=0}^{N-1}\frac{\partial b_n}{\partial S_i}\frac{\partial y}{\partial b_n} &= -\frac{1}{2}\frac{\partial y}{\partial b_{N-1}}\sum_{i=1}^{N}\frac{\partial b_{N-1}}{\partial S_i} + \text{holomorphic} \\ &= \frac{Nx^{N-1}}{y} + \text{holomorphic} \end{aligned} \tag{8.12}$$

we see that this expression has simple poles at P and Q with residues $\pm N$, and no other poles. The properties (8.11) and (8.12) suffice to show that

$$-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial y}{\partial S_i} = \frac{W_0''(x)}{y} \tag{8.13}$$

as the function on the r.h.s. has the same properties.

[9]The relation to the formulæ in ref. [32] is: $(1/M)\omega'(x) = -(1/M^2)\langle {\rm tr}(1/(x-\Phi))\,{\rm tr}(1/(x-\Phi))\rangle_{\rm conn}$.

To simplify the remainder of the discussion, we consider $N_f < N$. In this case, we found in the previous section that $f(x) \propto \prod_{I=1}^{N_f}(x+m_I)$, so $f(-m_I)=0$. The contours in eq. (8.8) are on the sheet on which $y=+W_0'(x)+\cdots$, and on this sheet, eqs. (7.10) and (7.11) imply $\omega_{\rm s}(-m_I)=0$ so this term drops out of eq. (8.7), yielding

$$\omega_{\rm d}(x) = -\frac{y-W_0'(x)}{2y}\sum_{I=1}^{N_f}\frac{1}{x+m_I} = -\frac{y-W_0'(x)}{2y}\frac{f'}{f}\,. \tag{8.14}$$

Collecting the above results one finds

$$\lambda_{SW} = \frac{x}{y}\left[W_0''(x) - \frac{1}{2}(W_0'(x)-y)\frac{f'}{f}\right] \tag{8.15}$$

which is in perfect agreement with the $N_f < N$ result in ref. [25].

9. Summary

In this paper we have continued the program initiated in [5] for analyzing $\mathcal{N}=2$ gauge theories within the matrix model approach [1]–[4]; here we included matter in the fundamental representation of $\mathrm{U}(N)$. This addition exposes new features of the method, one of which is the appearance of disk diagrams that contribute to the free energy. Similarly, the tadpole diagrams necessary for computing the periods a_i also have a contribution from disk diagrams. We computed the relation between a_i and the classical moduli e_i, as well as the $\mathcal{N}=2$ prepotential $\mathcal{F}(a)$, finding complete agreement with known results.

An interesting feature of our calculation is that the two cases $N_f < N$ and $N \le N_f < 2N$ are on the same footing and can be treated using the same method within the matrix model approach. The only difference between the two cases is the appearance of the polynomial $\tilde{T}(x)$ when $N_f \ge N$, cf. (4.10). In the final expression for the prepotential, however, $\tilde{T}(x)$ disappears. In section 6 we discussed the meaning of $\tilde{T}(x)$, explaining how it affects the form of the Seiberg-Witten curve when $N_f \ge N$.

From the point of view of computational efficiency, the matrix model approach cannot, in its present form, compete with other methods for computing multi-instanton contributions [34]–[36]. However, it would be interesting to connect these approaches with the matrix model perspective to improve our understanding of multi-instanton effects.

In sections 7 and 8 we presented derivations, entirely within the context of the matrix model, of the Seiberg-Witten curve and differential for the $\mathcal{N}=2$ $\mathrm{U}(N)$ theory with $N_f < 2N$ flavors. The contribution to the free energy from disk diagrams (7.20) played an important role in the analysis. A comparison of (7.24) and (7.29) exhibits the difference between the Seiberg-Witten curves for $N_f < N$ and $N \le N_f < 2N$. In the latter case, the matrix model makes a specific choice for the modification of the curve. This result was also inferred in section 6 from the perturbative calculation.

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